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Orbit Averaged Variation of Earth Satellite Orbital Elements

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ABSTRACT

An orbit averaging numerical integration method for determining the variation in shape and orientation of an earth satellite orbit is discussed.

Accepted for the Air Force
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ORBIT AVERAGED VARIATION OF EARTH SATELLITE ORBITAL ELEMENTS

I. INTRODUCTION

In this note we sketch the derivation of the equations for the osculating elliptic orbital elements of an earth satellite in terms of an angular independent variable and discuss an orbit averaging technique of integrating the equations. We employ equations which are valid for small eccentricity and inclination.

Because we are interested in earth satellite orbits for which moon perturbations are difficult to handle analytically, we discuss the numerical integration of the orbit averaged equations.

Gaussian quadrature of orbit averaged definite integrals is much faster in computer time than the exact integration of the equations of motion. Closed form analytic formulas for the definite integrals would be even faster to evaluate, but as was said the derivation of such formulas is extremely intricate for far out satellites.

The method of orbit averaging yields reasonably accurate predictions of the shape and orientation of the orbit, but not of the specific position in the orbit. A first order method of evaluating the orbit averaged definite integrals might cause the predictions to get out of phase with time. A second order method is described which uses a first order orbit averaged linear variation in the elements in evaluating the definite integrals. It would be more accurate to use a first order analytic non-orbit averaged variation in the elements in the second order evaluation of the definite integrals. Any such formulas would be easy to add to the computer program that is being written to implement the formulas in this note.

Explicit formulas for the perturbing accelerations due to the sun, moon and earth gravitational potential harmonics are presented in order to document the computer program as completely as possible.

II. CARTESIAN EQUATIONS OF MOTION

The equations of motion of an earth satellite are

$$\left.\begin{aligned} \frac{d^2 x^k}{dt^2} &= -\frac{\mu x^k}{r^3} + F^k \\ x^k &= x_o^k , \quad \frac{dx^k}{dt} = x_o^{k+3} \quad \text{when } t = t_o \end{aligned}\right\} k = 1, 2, 3 \tag{1}$$

where

x^k = k^{th} cartesian position component of satellite relative to earth $(k = 1, 2, 3)$

$\frac{dx^k}{dt}$ = x^{k+3} = k^{th} cartesian velocity component of satellite relative to earth $(k = 1, 2, 3)$

$$r = \sqrt{\sum_{j=1}^{3} (x^j)^3}$$

μ = gravitational constant times mass of earth (units are $L^3 T^{-2}$)

F^k = k^{th} component of acceleration on satellite additional to the central force acceleration $(k = 1, 2, 3)$

If the perturbing accelerations (F^1, F^2, F^3) were zero, the motion satisfying (1) would follow an elliptic trajectory. Even with non-zero perturbing accelerations there are elliptic elements associated with the postion and velocity $(x^1, \ldots, x^6)$ at time t, called the osculating elements. These are the elements of the elliptic orbit that the satellite would follow given the position

and velocity $(x^1, \ldots, x^6)$ at time t if the perturbing accelerations were turned off after time t.

The elliptic elements we shall employ are

a = Semi-major axis (distance unit consistent with the distance unit in μ)

e = Eccentricity $(0 \leq e < 1)$

I = Inclination $(0^\circ \leq I \leq 180^\circ)$

Ω = Right ascension of ascending node $(0^\circ \leq \Omega < 360^\circ)$

ω = Argument of perigee $(0^\circ \leq \omega < 360^\circ)$

M_o = Mean anomaly at time t_o $(0 \leq M_o < 360^\circ)$

See Ref. 1 or any book on celestial mechanics for the formulas for the position and velocity as a function of time in terms of the orbital elements and for the orbital elements in terms of the position and velocity at a given time. Additional notation for elliptic orbits which we shall employ is

p = $a(1 - e^2)$ = semi-latus rectum

n = $\mu^{1/2} a^{-3/2}$ = mean motion

M = $M_o + n(t - t_o)$ = mean anomaly at time t

ξ = eccentric anomaly

Ψ = true anomaly

η = $\omega + \Psi$ = angle measured along the orbital plane from the ascending node to the true position of the satellite

$\tilde{\omega}$ = $\omega + \Omega$ = longitude of perigee

$\tilde{\eta}$ = $\eta + \Omega$

In order to study the behavior of the osculating elements with time the cartesian equations of motion (1) can be numerically integrated and the osculating elements evaluated at each tabular point. However, numerical integration of (1) takes a fair amount of computer time and becomes prohibitive when studying a large variety of orbits.

III. EQUATIONS FOR OSCULATING ELEMENTS

Equations of motion (1) can be written in the form

$$\left.\begin{aligned} \frac{dx^k}{dt} &= x^{k+3} \\ \frac{dx^{k+3}}{dt} &= -\frac{\mu x^k}{r^3} + F^k \\ x^k = x_o^k \,,\quad x^{k+3} &= x_o^{k+3} \quad \text{when } t = t_o \end{aligned}\right\} k = 1,\ 2,\ 3 \tag{2}$$

Let $(\beta^1, \beta^2, \beta^3, \beta^4, \beta^5, \beta^6) = (a, e, I, \Omega, \omega, M_o)$ be the osculating elliptic orbital elements at time t with the functional relationship

$$x^k = x^k(\beta^1, \ldots, \beta^6, t) \;, \quad k = 1, \ldots, 6 \tag{3}$$

being given by the usual elliptic orbit formulas in such a way that

$$\left.\begin{aligned} \frac{\partial x^k}{\partial t} &= x^{k+3} \\ \frac{\partial x^{k+3}}{\partial t} &= -\frac{\mu x^k}{r^3} \end{aligned}\right\} \quad k = 1,\ 2,\ 3 \tag{4}$$

We have

$$\frac{dx^k}{dt} = \sum_{j=1}^{6} \frac{\partial x^k}{\partial \beta^j} \frac{d\beta^j}{dt} + \frac{\partial x^k}{\partial t} \;, \quad k = 1, \ldots, 6 \tag{5}$$

Equations (2), (3), (4) imply

$$\left.\begin{aligned} \sum_{j=1}^{6} \frac{\partial x^k}{\partial \beta^j} \frac{d\beta^j}{dt} &= 0 \\ \sum_{j=1}^{6} \frac{\partial x^{k+3}}{\partial \beta^j} \frac{d\beta^j}{dt} &= F^k \end{aligned}\right\} \quad k = 1,\ 2,\ 3 \tag{6}$$

We introduce the Lagrange brackets

$$[\beta^i , \beta^j] = \sum_{k=1}^{3} \left(\frac{\partial x^k}{\partial \beta^i} \frac{\partial x^{k+3}}{\partial \beta^j} - \frac{\partial x^{k+3}}{\partial \beta^i} \frac{\partial x^k}{\partial \beta^j} \right)$$

$$i,\ j = 1, \ldots, 6 \tag{7}$$

For a given value of i (between 1 and 6) multiply the first equation in (6) by $-\dfrac{\partial x^{k+3}}{\partial \beta^i}$ and the second equation in (6) by $\dfrac{\partial x^k}{\partial \beta^i}$ for $k = 1, 2, 3$ and then add the resulting six equations. We obtain

$$\sum_{j=1}^{6} [\beta^i , \beta^j] \frac{d\beta^j}{dt} = \sum_{k=1}^{3} F^k \frac{\partial x^k}{\partial \beta^i} \qquad i = 1, \ldots, 6 \tag{8}$$

The Lagrange brackets $[\beta^i , \beta^j]$ are evaluated and equations (8) solved for $d\beta^j/dt$ ($j = 1, \ldots, 6$) in most books on celestial mechanics; see, for example, Ref. 2, pages 273-307. We shall just give the final result.

Let $\vec{\epsilon}_r$ be a unit vector at the satellite pointing from the earth to the satellite; let $\vec{\epsilon}_s$ be a unit vector normal to $\vec{\epsilon}_r$ in the plane of the instantaneous position and velocity vectors of the satellite and making an acute angle with the velocity vector; and let $\vec{\epsilon}_w = \vec{\epsilon}_r \times \vec{\epsilon}_s$. Denote the components of the

acceleration (F^1, F^2, F^3) in the $\vec{\varepsilon}_r$, $\vec{\varepsilon}_s$ and $\vec{\varepsilon}_w$ directions by R, S and W, respectively. Then Gauss' form for the equations for the osculating elements are

$$
\left.
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}}\left[R\, e \sin\Psi + S\,\frac{p}{r}\right] \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na}\left[R \sin\Psi + S(\cos\xi + \cos\Psi)\right] \\
\frac{dI}{dt} &= \frac{rW}{na^2\sqrt{1-e^2}} \cos(\omega + \Psi) \\
\sin I\, \frac{d\Omega}{dt} &= \frac{rW}{na^2\sqrt{1-e^2}} \sin(\omega + \Psi) \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{nae}\left[-R\cos\Psi + S\left(\frac{r}{p} + 1\right)\sin\Psi\right] \\
&\qquad - \cos I\, \frac{d\Omega}{dt} \\
\frac{dM}{dt} &= n + R\left[-\frac{2r}{na^2} + \frac{1-e^2}{nae}\cos\Psi\right] \\
&\qquad - S\,\frac{\sin\Psi}{nae}\left[1 - e^2 + \frac{r}{a}\right]
\end{aligned}
\right\} \quad (9)
$$

The equation for the initial mean anomaly M_o is easily derived from the above if M_o rather than M is regarded as the primary orbital element.

IV. CHANGE OF INDEPENDENT VARIABLE

The eccentric anomaly ξ, true anomaly Ψ and mean anomaly M in an elliptic orbit are related by

$$\tan\frac{\Psi}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{\xi}{2}$$

$$M = \xi - e\sin\xi$$

From these formulas and equations (9) we find after some very intricate elliptic orbit computations

$$\frac{d\eta}{dt} = \frac{d\Psi}{dt} + \frac{d\omega}{dt} = \frac{(\mu p)^{1/2}}{r^2}\left(1 - \frac{r^3}{\mu p}\cot I\,\sin\eta\cdot W\right)$$

For the compound angular quantity $\tilde{\eta} = \eta + \Omega$ we then have

$$\frac{d\tilde{\eta}}{dt} = \frac{(\mu p)^{1/2}}{r^2} + \frac{r\sin\eta\cdot W}{(\mu p)^{1/2}}\,\frac{1-\cos I}{\sin I}$$

We define

$$\varkappa = \frac{1}{1 + \dfrac{r^3}{\mu p}\,\dfrac{1-\cos I}{\sin I}\,\sin\eta\cdot W} \tag{10}$$

Expressing equation (9) in terms of the independent variable $\tilde{\eta}$ instead of t we obtain (see Ref. 3)

$$\left.\begin{aligned}
\frac{da}{d\tilde{\eta}} &= \frac{2a^2 r^2 \varkappa}{\mu p} \{R\, e \sin\Psi + S(1 + e\cos\Psi)\} \\
\frac{de}{d\tilde{\eta}} &= \frac{r^2 \varkappa}{\mu} \{R \sin\Psi + \frac{r}{p}\,[2\cos\Psi + e(1 + \cos^2\Psi)] \cdot S\} \\
\frac{dI}{d\tilde{\eta}} &= \frac{r^3 \varkappa}{\mu p} \cos\eta \cdot W \\
\frac{d\Omega}{d\tilde{\eta}} &= \frac{r^3 \varkappa}{\mu p}\, \frac{\sin\eta}{\sin I} \cdot W \\
\frac{d\omega}{d\tilde{\eta}} &= \frac{r^2 \varkappa}{\mu e} \{-R\cos\Psi + S \cdot (1 + \frac{r}{p}) \sin\Psi\} - \cos I\, \frac{d\Omega}{d\tilde{\eta}} \\
\frac{dt}{d\tilde{\eta}} &= \frac{r^2 \varkappa}{(\mu p)^{1/2}}
\end{aligned}\right\} \quad (11)$$

We could replace the equation for the semi-major axis a with that for the semi-latus rectum p:

$$\frac{dp}{d\tilde{\eta}} = \frac{2r^3 \varkappa}{\mu} S \qquad (12)$$

V. SMALL ECCENTRICITY AND INCLINATION

Equations (11) are singular when $e = 0$. Therefore as in Ref. 2, p. 287, we introduce the variables

$$\begin{aligned} H &= e \sin\tilde{\omega} \\ K &= e \cos\tilde{\omega} \end{aligned} \qquad (13)$$

and replace the equations in (11) for e and ω by

$$\left.\begin{aligned}
\frac{dH}{d\tilde{\eta}} &= \frac{r^2\varkappa}{\mu}\Big\{-R\cos\tilde{\eta} + S\frac{r}{p}H(1+\cos^2\Psi) \\
&\quad + 2S\frac{r}{p}\sin\tilde{\eta} + S\left(1-\frac{r}{p}\right)\sin\Psi\cos\tilde{\omega}\Big\} + K(1-\cos I)\frac{d\Omega}{d\tilde{\eta}} \\
\frac{dK}{d\tilde{\eta}} &= \frac{r^2\varkappa}{\mu}\Big\{R\sin\tilde{\eta} + S\frac{r}{p}K(1+\cos^2\Psi) \\
&\quad + 2S\frac{r}{p}\cos\tilde{\eta} - S\left(1-\frac{r}{p}\right)\sin\Psi\sin\tilde{\omega}\Big\} - H(1-\cos I)\frac{d\Omega}{d\tilde{\eta}}
\end{aligned}\right\} \quad (14)$$

We used the variable $\tilde{\omega}$ instead of ω in (13) to ameliorate the singularity that occurs when $I = 0^{\circ}$ or 180°, which is also why we used $\tilde{\eta}$ instead of η as the independent variable. If I is not near 0° or 180°, say $10^{\circ} < I < 170^{\circ}$, we can employ the equations in (11) for I and Ω. However, if I is near 0° or 180°, we follow Ref. 2, p. 288 and make the change of variables.

$$\begin{aligned}
P &= \tan I \sin\Omega \\
Q &= \tan I \cos\Omega
\end{aligned} \quad (15)$$

and replace the equations in (11) for I and Ω by

$$\left.\begin{aligned}
\frac{dP}{d\tilde{\eta}} &= \frac{r^3\varkappa}{\mu p} W\left[\frac{\sin\tilde{\eta} + (\cos I - 1)\cos\Omega\sin\eta}{\cos^2 I}\right] \\
\frac{dQ}{d\tilde{\eta}} &= \frac{r^3\varkappa}{\mu p} W\left[\frac{\cos\tilde{\eta} + (1-\cos I)\sin\Omega\sin\eta}{\cos^2 I}\right]
\end{aligned}\right\} \quad (16)$$

We cannot entirely replace I, Ω by P, Q because P, Q and the equations they satisfy are singular when $I = 90^{\circ}$.

VI. ORBIT AVERAGED VARIATION

Changing notation, let

$$\left.\begin{aligned}
\beta^1 &= a \\
\beta^2 &= H \\
\beta^3 &= K \\
\beta^4 &= \begin{cases} P & \text{if } I \text{ near } 0^\circ \text{ or } 180^\circ \\ I & \text{otherwise} \end{cases} \\
\beta^5 &= \begin{cases} Q & \text{if } I \text{ near } 0^\circ \text{ or } 180^\circ \\ \Omega & \text{otherwise} \end{cases}
\end{aligned}\right\} \tag{17}$$

Then equations (11), (14) and (16) can be written in the symbolic form

$$\left.\begin{aligned}
\frac{d\beta^k}{d\tilde{\eta}} &= f^k(\beta^1, \ldots, \beta^5, \tilde{\eta}, t) \qquad k = 1, \ldots, 5 \\
\frac{dt}{d\tilde{\eta}} &= f^6(\beta^1, \ldots, \beta^5, \tilde{\eta}, t)
\end{aligned}\right\} \tag{18}$$

Exact integration of these equations would give the exact motion of the body.

The osculating orbital elements are slowly varying functions of time, unlike the cartesian coordinates. As a function of $\tilde{\eta}$, the behavior of an osculating element might be as in Fig. 1. During one revolution ($\tilde{\eta}$ increasing by 2π) the value of an element oscillates; but from one revolution to the next there is a secular or long periodic trend. If it is sufficient to know this mean behavior of the orbital elements rather than their exact behavior, we can follow Ref. 3 and use a method of orbit averaging which results in considerable savings in computer time.

Namely, in a first order theory we hold the elements $(\beta^1, \ldots, \beta^5)$ constant in the right side of (18) and evaluate t from $\tilde{\eta}$ using these elements and elliptic orbit formulas. Then the first order mean changes in the elements during revolution j are

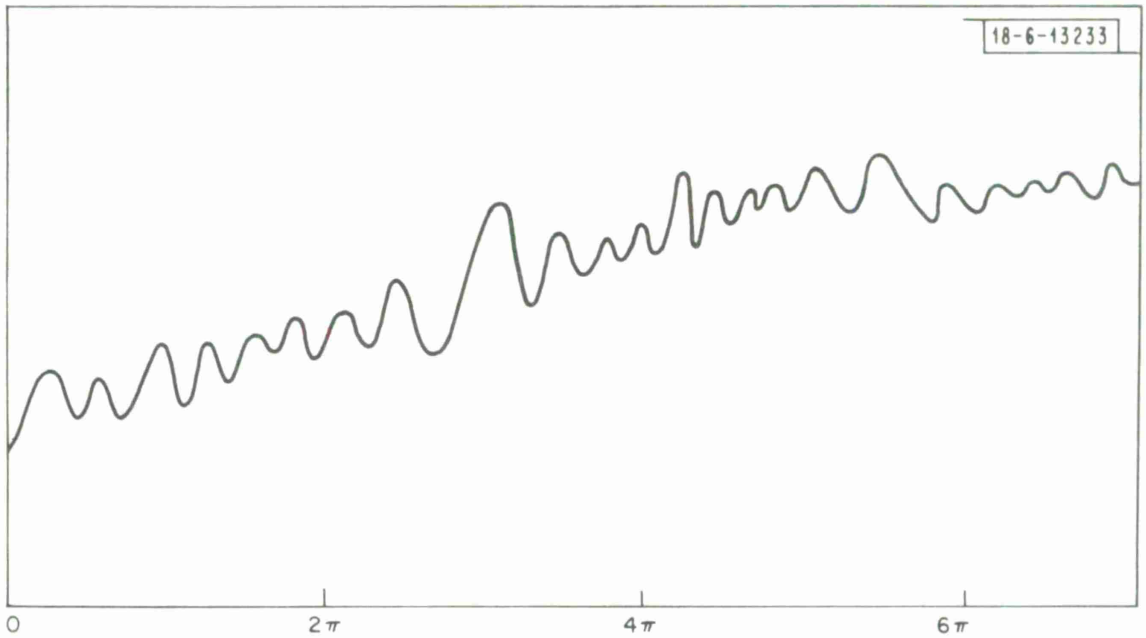

Fig. 1. Possible behavior of an osculating elliptic orbital element as a function of the angular variable $\tilde{\eta}$.

$$
\left.
\begin{aligned}
\Delta\beta_j^k &= \int_{\tilde\eta_1}^{\tilde\eta_2} f^k(\beta^1, \ldots, \beta^5, \tilde\eta, t)\, d\tilde\eta\,, \quad k = 1, \ldots, 5 \\
\text{period} = \Delta t_j &= \int_{\tilde\eta_1}^{\tilde\eta_2} f^6(\beta^1, \ldots, \beta^5, \tilde\eta, t)\, d\tilde\eta
\end{aligned}
\right\} \qquad (19)
$$

where $\tilde\eta_1$ is the value of $\tilde\eta$ for which an orbit is considered to commence, where $\tilde\eta_2 = \tilde\eta_1 + 2\pi$ and when $\beta^k = \beta^k_{j-1}$, the value of β^k at the end of the previous revolution (j-1) .

We can evaluate the integrals in (19) to second order by taking

$$
\beta^k = \beta^k_{j-1} + \frac{(\tilde\eta - \tilde\eta_1)}{2\pi} \Delta\beta^k_{j-1} \quad , \quad k = 1, \ldots, 5 \qquad (20)
$$

where $\Delta\beta^k_{j-1}$ is the variation in the elements extrapolated from one or more immediately preceding revolutions. The first revolution for which there is no previous revolution can be integrated twice, the first time with the orbital elements constant and the second time with the variation in the orbital elements calculated in the first integration. Thereafter, each revolution need only be integrated once with the variation in the elements in the integrands being extrapolated from the variation determined in one or more previous revolutions.

Instead of using the orbit averaged first order linear variation of the elements in the second order integration, better accuracy could be obtained by using non-orbit averaged first order variations in the elements. These would have to be derived analytically.

It is desirable to use a second order evaluation of the integrals (19) in order to calculate the period accurately from one orbit to the next. However, the accurate calculation of the secular and long period behavior of the elements $(\beta^1, \ldots, \beta^5)$ giving the shape and orientation of the orbit are not so dependent on the use of a second order theory, except insofar as their prediction might get out of phase with time.

Our main concern is with earth satellite orbits for which the moon moves appreciably during an orbital revolution and the ratio of the distances of the satellite and the moon from the earth is not small. These

facts cause difficulties with analytic treatments. Therefore, the doctrine expounded in this note is to evaluate the integrals in (19) numerically. An accurate numerical second order orbit averaged theory might still require an analytic non-orbit averaged first order theory instead of the first order theory given by (20). It would be easy to add such analytic expressions to a computer program which numerically evaluates the integrals (19). We therefore do not persue the possibilities of analytic non-orbit averaged first order theories in this note.

A computer program would run faster with completely closed analytic formulas than it would if the integrals in (19) were evaluated numerically. However, the numerical integration of the definite integrals in (19) is still faster than numerically integrating exact equations of motion (1). Since the cartesian coordinates enter implicitely in the right side of (1) a predictor-corrector numerical integration method is employed and from 100 to 200 or more steps made per orbital revolution. The number of steps cannot be decreased even if less accuracy is required because the integration becomes unstable. On the other hand, everything in the integrands in (19) are explicite functions of the independent variable. Therefore, an integration technique such as Gaussian quadrature can be employed with many fewer steps per orbital revolution.

VII. ELLIPTIC ORBIT FORMULAS

Given values of $(\beta^1, \ldots, \beta^5)$ and $\tilde{\eta}$ we must evaluate the various quantities which enter in the numerical integration of the right sides of (19). Of course, $\beta^1 = a$. By (13) and (17) we have

$$e = \left[(\beta^2)^2 + (\beta^3)^2\right]^{1/2} \tag{21}$$

If $e = 0$ we take $\omega = 0$ and $\omega = \Omega$ by convention. If $e > 0$ we have

$$\begin{aligned} \sin \tilde{\omega} &= \beta^2/e \\ \cos \tilde{\omega} &= \beta^3/e \\ \omega &= \tilde{\omega} - \Omega \end{aligned} \tag{22}$$

We either have $I = \beta^4$, $\Omega = \beta^5$ or by (15)

$$\tan I = \left[(\beta^4)^2 + (\beta^5)^2\right]^{1/2}$$
$$\sin \Omega = \beta^4/\tan I \qquad (23)$$
$$\cos \Omega = \beta^5/\tan I$$

We then calculate

$$\Psi = \tilde{\eta} - \tilde{\omega}$$
$$\eta = \tilde{\eta} - \Omega$$
$$p = a(1 - e^2) \qquad (24)$$
$$n = \mu^{1/2} a^{-3/2}$$
$$r = \frac{p}{1 + e \cos \Psi}$$

We calculate the transformation matrix

$$B_{11} = \cos \Omega \cos \eta - \sin \Omega \sin \eta \cos I$$
$$B_{12} = -\cos \Omega \sin \eta - \sin \Omega \cos \eta \cos I$$
$$B_{13} = \sin \Omega \sin I$$
$$B_{21} = \sin \Omega \cos \eta + \cos \Omega \sin \eta \cos I$$
$$B_{22} = -\sin \Omega \sin \eta + \cos \Omega \cos \eta \cos I \qquad (25)$$
$$B_{23} = -\cos \Omega \sin I$$
$$B_{31} = \sin \eta \sin I$$
$$B_{32} = \cos \eta \sin I$$
$$B_{33} = \cos I$$

Then the relations between the unit vectors $\vec{\epsilon}_1$, $\vec{\epsilon}_2$, $\vec{\epsilon}_3$ in the x^1, x^2, x^3 directions and the unit vectors $\vec{\epsilon}_r$, $\vec{\epsilon}_s$, $\vec{\epsilon}_w$ defined in Section III are

$$\vec{\epsilon}_r = \sum_{j=1}^{3} B_{j1} \vec{\epsilon}_j$$

$$\vec{\epsilon}_s = \sum_{j=1}^{3} B_{j2} \vec{\epsilon}_j \tag{26}$$

$$\vec{\epsilon}_w = \sum_{j=1}^{3} B_{j3} \vec{\epsilon}_j$$

and the cartesian coordinates of the satellite relative to the earth are

$$x^k = rB_{k1} \quad , \quad k = 1, 2, 3 \tag{27}$$

To calculate the time t from the epoch when $\tilde{\eta}$ was $\tilde{\eta}_1$, we perform the following steps:

$$\xi = 2 \tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\psi}{2}\right)$$

$$\xi_o = 2 \tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan \left(\frac{\tilde{\eta}_1 - \tilde{\varpi}}{2}\right)\right)$$

$$M_o = \xi_o - e \sin \xi_o \tag{28}$$

$$M = \xi - e \sin \xi$$

$$t = \frac{1}{n} (M - M_o)$$

The next two sections derive the expressions for the acceleration on the satellite

$$\vec{F} = F^1 \vec{\epsilon}_1 + F^2 \vec{\epsilon}_2 + F^3 \vec{\epsilon}_3 \tag{29}$$

due to the sun, moon and earth gravitational potential harmonics at a given

time. The components of $\vec{F}$ in the $\vec{\varepsilon}_r$, $\vec{\varepsilon}_s$ and $\vec{\varepsilon}_w$ directions are then

$$\begin{aligned} R &= \vec{F} \cdot \vec{e}_r \\ S &= \vec{F} \cdot \vec{e}_s \\ W &= \vec{F} \cdot \vec{e}_w \end{aligned} \tag{30}$$

VIII. ACCELERATION DUE TO THE SUN AND MOON

We define

x^k_{me} = k^{th} coordinate of the moon relative to the earth ($k = 1, 2, 3$)

x^k_{cs} = k^{th} coordinate of the earth-moon barycenter relative to the sun ($k = 1, 2, 3$)

x^k_{se} = k^{th} coordinate of the sun relative to the earth ($k = 1, 2, 3$)

M_s = mass of sun

M_m = mass of moon

M_e = mass of earth

M_c = $M_e + M_m$

γ = gravitational constant (so that $\mu = \gamma M_e$)

We have

$$x^k_{se} = -\left(x^k_{cs} - \frac{M_m}{M_c}\, x^k_{me}\right) \quad , \quad k = 1, 2, 3 \tag{31}$$

Given the time we read a magnetic tape and interpolate to determine x^k_{me} , x^k_{cs} $(k = 1, 2, 3)$. The already existing magnetic tape and computer subroutines which we possess give the coordinates of the moon and sun in the coordinate system referred to the mean equinox and equator of 1950.0 . We have so far left the coordinate system in which we integrate the equations of motion unspecified. Given the subroutines which already exist, we shall use the inertial system referred to the mean equinox and equator of 1950.0 .

Having determined the position of the sun and moon relative to the earth, the perturbing accelerations due to these bodies on the motion of the satellite relative to the earth are the differences in the accelerations on the satellite and earth:

$$F^k = \gamma \sum_{\alpha = m, s} M_\alpha \left[\frac{x^k_{\alpha b}}{r^3_{\alpha b}} - \frac{x^k_{\alpha e}}{r^3_{\alpha e}} \right] , \quad k = 1, 2, 3 \tag{32}$$

where for $\alpha = m, s$ and $\lambda = b, e$

$$x^k_{\alpha b} = x^k_{\alpha e} - x^k , \quad k = 1, 2, 3$$

$$r_{\alpha\lambda} = \left[\sum_{j=1}^{3} (x^j_{\alpha\lambda})^2 \right]^{1/2}$$

IX. ACCELERATION DUE TO EARTH GRAVITATIONAL POTENTIAL HARMONICS

Let (y^1, y^2, y^3) be a coordinate system with origin at the center of mass of the earth, with y^3 axis pointing along the axis of rotation of the earth, with y^1 axis normal to y^3 lying in the plane which contains the y^3 axis and Greenwich, and with y^2 axis completing the right hand system. The relation between this coordinate system and the one (x^1, x^2, x^3) referred to the mean equinox and equator of 1950.0 is

$$\left.\begin{aligned} y^k &= \sum_{\ell=1}^{3} A_{k\ell}\, x^\ell \\ x^k &= \sum_{\ell=1}^{3} A_{\ell k}\, y^\ell \end{aligned}\right\} \quad k = 1, 2, 3 \tag{33}$$

where the orthogonal rotation-nutation-precession matrix (A_{ij}) is discussed in Section X.

We introduce polar coordinates (r, θ, ϕ) rotating with the earth by

$$\left.\begin{aligned} y^1 &= r \cos\theta \cos\phi & \qquad 0 &\le r < \infty \\ y^2 &= r \sin\theta \cos\phi & \qquad 0 &\le \theta < 2\pi \\ y^3 &= r \sin\phi & \qquad -\frac{\pi}{2} &\le \phi \le \frac{\pi}{2} \end{aligned}\right\} \tag{34}$$

We have

$$r^2 = \sum_{\ell=1}^{3} (x^\ell)^2 = \sum_{\ell=1}^{3} (y^\ell)^2 \tag{35}$$

$$\left.\begin{aligned} \sin\phi &= \frac{1}{r} \sum_{\ell=1}^{3} A_{3\ell}\, x^\ell \\ \cos\phi &= +\sqrt{1 - \sin^2\phi} \end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} \cos\theta &= \frac{1}{r\cos\phi} \sum_{\ell=1}^{3} A_{1\ell}\, x^\ell \\ \sin\theta &= \frac{1}{r\cos\phi} \sum_{\ell=1}^{3} A_{2\ell}\, x^\ell \end{aligned}\right\} \tag{37}$$

Outside the earth the gravitational potential of the earth can be expressed in spherical harmonics by

$$U = -\frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty} \frac{J_n}{r^n} P_n(\sin\phi)\right.$$

$$+ \sum_{n=2}^{\infty} \sum_{h=1}^{n} \frac{1}{r^n}\left[C_{nh}\cos h\,\theta + S_{nh}\sin h\,\theta\right]$$

$$\left.\cdot\; P_{nh}(\sin\phi)\right\} \tag{38}$$

where P_n and P_{nh} are the Legendre ploynomials and generalized Legendre functions, respectively (see Section XI). The summation starts with $n = 2$ rather than with $n = 1$ because the origin of the coordinate system is at the center of mass of the earth.

Since to a high degree of approximation the y^3 axis is a principal moment of inertia axis, we have

$$C_{21} = 0 \;,\; S_{21} = 0 \tag{39}$$

The J_n are called the zonal harmonic coefficients and the C_{nh}, S_{nh} are called the tesseral harmonic cosine and sine coefficients. C_{nn} and S_{nn} are also known as sectorial harmonic coefficients. The sign and notation conventions are those adopted by the Smithsonian Astrophysical Observatory; see Ref. 4.

The cosines and sines of multiples of the longitude θ can be calculated by

$$\left.\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
&\;\;\vdots \\
\cos h\theta &= \cos(h-1)\theta\cos\theta - \sin(h-1)\theta\sin\theta \\
\sin h\theta &= \sin(h-1)\theta\cos\theta + \cos(h-1)\theta\sin\theta
\end{aligned}\right\} \tag{40}$$

The acceleration on the satellite due to the gravitational attraction of the earth is $-\operatorname{grad} U$. Thus, the k^{th} component of the acceleration in the (x^1, x^2, x^3) coordinate system is

$$F^k = -\frac{\partial U}{\partial x^k} = -\frac{\mu x^k}{r^3} + \mu \sum_{n=2}^{\infty} \frac{J_n}{r^{n+2}} \left[\frac{(n+1)x^k}{r} P_n(\sin\phi) - P_n'(\sin\phi)\, r \frac{\partial \sin\phi}{\partial x^k}\right]$$

$$+ \mu \sum_{n=2}^{\infty} \sum_{h=1}^{n} \frac{1}{r^{n+2}} \left\{\left[C_{nh} \cos h\theta + S_{nh} \sin h\theta\right] \cdot \left[-\frac{(n+1)x^k}{r} P_{nh}(\sin\phi)\right.\right.$$

$$\left. + P_{nh}'(\sin\phi)\, r \frac{\partial \sin\phi}{\partial x^k}\right] + h\left[-C_{nh} \sin h\theta + S_{nh} \cos h\theta\right]$$

$$\left. \cdot P_{nh}(\sin\phi) \cdot r \frac{\partial\theta}{\partial x^k}\right\} \tag{41}$$

where by (36)

$$r \frac{\partial \sin\phi}{\partial x^k} = A_{3k} - \frac{x^k}{r} \sin\phi \tag{42}$$

and by (37)

$$-\sin\theta \frac{\partial\theta}{\partial x^k} = \frac{1}{r\cos\phi} A_{1k} - \frac{1}{\cos\phi} \frac{\partial \cos\phi}{\partial x^k} \cos\theta - \frac{x^k}{r^2} \cos\theta$$

$$\cos\theta \frac{\partial\theta}{\partial x^k} = \frac{1}{r\cos\phi} A_{2k} - \frac{1}{\cos\phi} \frac{\partial \cos\phi}{\partial x^k} \sin\theta - \frac{x^k}{r^2} \sin\theta$$

Multiplying the first equation by $-\sin\theta$ and the second by $\cos\theta$ and adding we obtain

$$r \frac{\partial\theta}{\partial x^k} = \frac{1}{\cos\phi} \left[A_{2k} \cos\theta - A_{1k} \sin\theta\right] \tag{43}$$

For use in (1) and (30) the $-\mu x^k/r^3$ term should be dropped from (41). The order of the harmonics to be employed depends on the case being considered. For all satellites it is necessary to include J_2. For most satellites the inclusion of a few additional zonals would be sufficient. For a synchronous satellite we would certainly want to include C_{22} and S_{22}.

X. ROTATION – NUTATION – PRECESSION OF THE EARTH

The transformation matrix (A_{ij}) appearing in (33) is defined by

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = SNP \tag{44}$$

where S gives the transformation between coordinates (y^1, y^2, y^3) rotating with the earth and those referred to the true equinox and equator of date, where N gives the transformation between coordinates referred to the true equinox and equator of date and those referred to the mean equinox and equator of date, and where P gives the transformation between coordinates referred to the mean equinox and equator of date and those (x^1, x^2, x^3) referred to the mean equinox and equator of 1950.0. Matrix multiplication follows the usual row x column rule, so that, for instance,

$$(NP)_{ij} = \sum_{k=1}^{3} N_{ik} P_{kj} \tag{45}$$

Following Ref. 5, p. 482, we define the angles

$$\begin{aligned} \zeta_o &= 2304''\!.948T + 0''\!.302T^2 + 0''\!.0179T^3 \\ z &= 2304''\!.948T + 1''\!.093T^2 + 0''\!.0192T^3 \\ \theta &= 2004''\!.255T - 0''\!.426T^2 - 0''\!.0416T^3 \end{aligned} \tag{46}$$

where T is measured in tropical centuries of 36524.21988 ephemeris days from the epoch 1950.0 (JED 2433282.423) to the instant of interest. Then by Ref. 6, p. 31, the precession matrix P at this instant is

$$
\begin{aligned}
P_{11} &= \cos\zeta_o \cos\theta \cos z - \sin\zeta_o \sin z \\
P_{12} &= -\sin\zeta_o \cos\theta \cos z - \cos\zeta_o \sin z \\
P_{13} &= -\sin\theta \cos z \\
P_{21} &= \cos\zeta_o \cos\theta \sin z + \sin\zeta_o \cos z \\
P_{22} &= -\sin\zeta_o \cos\theta \sin z + \cos\zeta_o \cos z \\
P_{23} &= -\sin\theta \sin z \\
P_{31} &= \cos\zeta_o \sin\theta \\
P_{32} &= -\sin\zeta_o \sin\theta \\
P_{33} &= \cos\theta
\end{aligned}
\tag{47}
$$

Let τ denote the time from the epoch 1950.0 (JED 2433282.423) in units of 10,000 ephemeris days. Then by Taylor's theorem we have

$$
P_{jk} = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n P_{jk}}{d\tau^n} \right|_{\tau=0} \tau^n \quad , \qquad j, k = 1, 2, 3 \tag{48}
$$

Treating the coefficients in (46) as exact, some simple calculations show that the terms up to the fifth power in the Taylor expansions (48) are:

$$
\left.
\begin{aligned}
P_{11} &= 1.0 - 2.22603398052517 \times 10^{-5}\tau^{2} - 2.6903385325366 \times 10^{-9}\tau^{3} \\
&\quad + 8.191221606878 \times 10^{-11}\tau^{4} + 1.79948222850 \times 10^{-14}\tau^{5} \\
P_{12} &= -6.119064710033514 \times 10^{-3}\tau - 5.06975739290688 \times 10^{-7}\tau^{2} \\
&\quad + 4.5321716219079 \times 10^{-8}\tau^{3} + 8.619581795926 \times 10^{-12}\tau^{4} \\
&\quad - 1.02943658327 \times 10^{-13}\tau^{5} \\
P_{13} &= -2.660399722772102 \times 10^{-3}\tau + 1.54818397804898 \times 10^{-7}\tau^{2} \\
&\quad + 1.9729201591810 \times 10^{-8}\tau^{3} + 1.960730253191 \times 10^{-12}\tau^{4} \\
&\quad - 4.39298354075 \times 10^{-14}\tau^{5} \\
P_{21} &= 6.119064710033514 \times 10^{-3}\tau + 5.06975739290688 \times 10^{-7}\tau^{2} \\
&\quad - 4.5321716219079 \times 10^{-8}\tau^{3} - 9.636891635856 \times 10^{-12}\tau^{4} \\
&\quad + 1.02604298897 \times 10^{-13}\tau^{5} \\
P_{22} &= 1.0 - 1.87214764627888 \times 10^{-5}\tau^{2} - 3.1022173551368 \times 10^{-9}\tau^{3} \\
&\quad + 6.882478825535 \times 10^{-11}\tau^{4} + 1.91215207447 \times 10^{-14}\tau^{5} \\
P_{23} &= -8.13957902909886 \times 10^{-6}\tau^{2} - 5.8309700675934 \times 10^{-10}\tau^{3} \\
&\quad + 2.994360606802 \times 10^{-11}\tau^{4} + 5.71739459043 \times 10^{-15}\tau^{5} \\
P_{31} &= 2.660399722772102 \times 10^{-3}\tau - 1.54818397804898 \times 10^{-7}\tau^{2} \\
&\quad - 1.9729201591810 \times 10^{-8}\tau^{3} + 3.791379581151 \times 10^{-13}\tau^{4} \\
&\quad + 4.50404085077 \times 10^{-14}\tau^{5} \\
P_{32} &= -8.13957902909886 \times 10^{-6}\tau^{2} + 1.8168268497009 \times 10^{-10}\tau^{3} \\
&\quad + 3.024323052660 \times 10^{-11}\tau^{4} + 2.58550054981 \times 10^{-17}\tau^{5} \\
P_{33} &= 1.0 - 3.53886334246294 \times 10^{-6}\tau^{2} + 4.1187882260017 \times 10^{-10}\tau^{3} \\
&\quad + 1.308742781343 \times 10^{-11}\tau^{4} - 1.12669845971 \times 10^{-15}\tau^{5}
\end{aligned}
\right\} \quad (49)
$$

The mean obliquity of the ecliptic is (see Ref. 6, p. 98)

$$
\varepsilon_{o} = 23^{\circ}27'08''.26 - 46''.845T - 0''.0059T^{2} + 0''.00181T^{3} \qquad (50)
$$

where T is measured in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 E.T. = JED 2415020.0 to the instant of interest. Let $\Delta\Psi$ and $\Delta\epsilon$ be the nutations in longitude and obliquity, respectively, as given by the series in Ref. 6, p. 44-45. The true obliquity of the ecliptic is then

$$\epsilon = \epsilon_o + \Delta\epsilon \quad . \tag{51}$$

The nutation matrix is given by (see Ref. 6, p. 43)

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} = \begin{bmatrix} 1 & -\Delta\Psi\cos\epsilon & -\Delta\Psi\sin\epsilon \\ \Delta\Psi\cos\epsilon & 1 & -\Delta\epsilon \\ \Delta\Psi\cos\epsilon & \Delta\epsilon & 1 \end{bmatrix} \tag{52}$$

Sidereal time is defined as the hour angle of the first point of Aires (equinox point). We therefore have

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{53}$$

where Θ is the Greenwich true sidereal time. By Ref. 6, p. 72,

$$\Theta = \Theta_o + \Delta\Psi\cos\epsilon \tag{54}$$

where Θ_o is the Greenwich mean sidereal time. By Ref. 5, p. 478, the Greenwich mean sidereal time $\bar{\Theta}_o$ at 0^h universal time on the day of interest is

$$\bar{\Theta}_o = 6^h 38^m 45^s\!.836 + 8,640,184^s\!.542T + 0^s\!.0929T^2 \tag{55}$$

where T denotes the number of Julian centuries of 36525 days which, at midnight beginning of day, have elapsed since mean noon on 1900 January 0 at the Greenwich meridian (Julian Date 2415020.0). The Greenwich mean sidereal time Θ_o at a given instant UT of universal time on that day is then

$$\Theta_o = \bar{\Theta}_o + \frac{d\Theta_o}{dt} \times UT \tag{56}$$

where by Ref. 6, pp. 75-76,

$$\frac{d\Theta_o}{dt} = (1.002737909265 + 0.589 \times 10^{-10}T)$$

$$\text{sidereal time seconds per universal time second} \tag{57}$$

The equations of motion are integrated as a function of Ephemeris Time ET, and the value of $ET - UT$ is given in Ref. 5, p. vii .

It was implicitely assumed in Section IX that the coordinate system (y^1, y^2, y^3) defined in terms of the axis of rotation of the earth is fixed in the crust of the earth. This is not exactly true and in fact if (44) is to define the transformation matrix for coordinates fixed in the earth to those referred to the mean equinox and equator of 1950.0 it should read $A = WSNP$. However, the wobble matrix W is so nearly the identity matrix that it can be ignored for the problem discussed in this note.

XI. LEGENDRE POLYNOMIALS AND FUNCTIONS

By Ref. 7, pp. 83 and 327, the definitions of the Legendre polynomials P_n and generalized Legendre functions P_{nh} are

$$\left.\begin{aligned} P_o(Z) &= 1 \\ P_n(Z) &= \frac{1}{2^n n!} \frac{d^n (Z^2 - 1)^n}{dZ^n} \qquad n = 1, 2, \ldots \end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned} P_{no}(Z) &= P_n(Z) \qquad n = 0, 1, 2, \ldots \\ P_{nh}(Z) &= (1 - Z^2)^{h/2} \frac{d^h}{dZ^h} P_n(Z) \quad , \quad h = 1, \ldots, n \end{aligned}\right\} \tag{59}$$

From these definitions it follows that

$$P_n(Z) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \frac{(2n-2i)\,!}{2^n (n-i)\,!\,(n-2i)\,!\,i\,!} Z^{n-2i} \tag{60}$$

$$P_{nh}(Z) = (1 - Z^2)^{h/2} \sum_{i=0}^{[\frac{n-h}{2}]} \frac{(2n-2i)\,!}{2^n (n-i)\,!\,(n-h-2i)\,!\,i\,!} Z^{n-h-2i} \tag{61}$$

where $[x]$ denotes the largest integer less than or equal to x . A computer subroutine to evaluate the Legendre polynomials and functions and their derivatives should use the various recursion formulas that exist.

In Ref. 4 expansion (38) is employed with $\overline{C}_{nh}$, $\overline{S}_{nh}$ and $\overline{P}_{nh}$ used in place of C_{nh} , S_{nh} and P_{nh} , where

$$\overline{P}_{nh} = \sqrt{\frac{2(2n+1)\,(n-h)\,!}{(n+h)\,!}} P_{nh} \tag{62}$$

This is because the integral of $\overline{P}_{nh}(\sin\phi)$ times $\cos h\theta$ or $\sin h\theta$ over the sphere is 4π . However, Ref. 4 does not use the corresponding normalization for the zonal harmonics

$$\overline{P}_n = \sqrt{2n+1}\; P_n \tag{63}$$

In using the values of J_n, $\overline{C}_{nh}$, $\overline{S}_{nh}$ given in Ref. 4 from fits to satellite data these different definitions should be taken into account.

XII. GAUSSIAN QUADRATURE

The method of numerical integration that should be used in evaluating the definite integrals (19) is Gaussian quadrature. Let $P_n(x)$ be the n^{th} Legendre polynomial. Following Ref. 8, pp. 319-325, we have

$$\int_{-1}^{1} f(y)\, dy = \sum_{i=1}^{m} H_i f(y_i) \tag{64}$$

where y_i is the i^{th} zero of $P_m(y)$ and where

$$H_i = \frac{2}{m\, P_{m-1}(y_i)\, P'_m(y_i)}$$

$$= \frac{2(1 - y_i^2)}{(m+1)^2\, [P_{m+1}(y_i)]^2}$$

The formula (64) is exact for polynomials f of order $2m - 1$. For the integrals of interest to us we have

$$\int_{\tilde{\eta}_1}^{\tilde{\eta}_2} f(\tilde{\eta})\, d\tilde{\eta} = \sum_{j=1}^{\ell} \int_{\frac{2\pi(j-1)}{\ell} + \tilde{\eta}_1}^{\frac{2\pi j}{\ell} + \tilde{\eta}_1} f(\tilde{\eta})\, d\tilde{\eta}$$

since $\tilde{\eta}_2 = \tilde{\eta}_1 + 2\pi$. To evaluate

$$\int_a^b f(\tilde{\eta})\, d\tilde{\eta}$$

we define

$$\tilde{\eta} = \left(\frac{b-a}{2}\right) y + \left(\frac{b+a}{2}\right)$$

$$d\tilde{\eta} = \left(\frac{b-a}{2}\right) dy$$

so that

$$\int_a^b f(\tilde{\eta})\, d\tilde{\eta} = \frac{b-a}{2} \int_{-1}^{1} f\left[\left(\frac{b-a}{2}\right) y + \left(\frac{b+a}{2}\right)\right] dy$$

$$= \frac{b-a}{2} \sum_{i=1}^{m} H_i f\left[\left(\frac{b-a}{2}\right) y_i + \left(\frac{b+a}{2}\right)\right]$$

Tables of y_i, $H_i (i = 1, \ldots, m)$ are given in Ref. 9 for $m = 1, \ldots, 16$. The values of ℓ and m should be input and chosen to give the desired accuracy in the fastest time.

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